Time - frequency approach to analysis of time varying dynamic systems

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Abstract — The paper discusses a problem of description and interpretation of time-varying systems behavior as observed in power system. Accurate power system identification allows for time-varying model system description, which better describes current/voltage relations in power system than classical linear time invariant (LTI) model.

The time variant active system description makes difficult its time-varying (t-v) model identification and implementation. The presented paper presents insight into properties of periodically varying systems illustrated with simple analog and discrete examples. The theoretical background and simulation results are aimed to help in understanding time variance as a new property of linear systems.

I. INTRODUCTION

There are many reasons for identifying the equivalent parameters of an electrical distribution system and its loads [1]. The knowledge of the parameters enables one to predict the system response to the compensating and harmonic suppressing equipment signals. The on-line measurement of these parameters may be necessary for an effective control of electronic compensators. Such measurement of the system parameters requires that the tested system has to be somehow disturbed. The disturbing signal should be measured and processed in order to obtain required data. This identification task is difficult, as the electrical system cannot be treated as strictly linear and time-invariant. Because of time-varying and nonlinear loads the system should be considered as nonlinear or time varying, even during one period of the system voltage.

The objective of the presented paper is to give some insight into behavior of the periodically time varying system. Frequency characteristics and impulse responses of such system are presented. The presented results were obtained via computer simulation and measurements in real electrical power.

II. PERIODICALLY VARYING DYNAMIC SYSTEMS

Owing to the increasing power of electronic loads connected to distribution systems, electrical power systems should be treated as nonlinear and time varying. For harmonic compensator control purposes the system can be modeled as linear but periodically varying. It follows from the fact, that fundamental harmonic is always dominated. The mathematical introduction given in this section concerns the local solution of nonlinear system with periodic input function. The short mathematical introduction presented below gives the general reason for above assumption.

Let us consider the system of differential equations in matrix form

\[ \frac{dz}{dt} = f(t, z) \]  

(1)

It is assumed that \( z \) is \( n \)-dimensional state vector and function \( f(x, t) \) on the right side of eqn. (1) is \( T \)-periodic, \( n \)-times differentiable in considered time interval.

Let \( w(t) \) be a given \( T \)-periodic solution of eqn. (1) and we seek the solution \( z(t) \) in the neighborhood of \( w(t) \) such that

\[ z(t) = w(t) + x(t) \]  

(2)

Substituting (1) into (2) and then expanding the right side of equation in Taylor series and omitting the terms with \( x \) of power two and higher, we obtain the linear system [2]

\[ \frac{dx}{dt} = A(t)x \]  

(3)

where matrix \( A(t) \) entries are periodical.

For such linear system it is possible to use such notions as frequency characteristics, characteristic numbers, characteristic exponents.

According to Floquet theorem the linear differential eqn. (3) with periodical matrix \( A(t) \) can be reduced to differential equation with constant coefficients

\[ \frac{dy}{dt} = By \]  

(4)

where \( B \) is a constant matrix, in general case with complex entries.

If matrix \( A(t) \) in eqn. (3) is real, \( T \)-periodic and the system solution is considered as periodic with period 2\( T \), then matrix \( B \) of associated equation (5) is always real. Constant matrix \( B \) can be found from matrix \( A(t) \) by using so called [1] fundamental matrix \( C \). For each periodic solution

\[ x = \Phi(t) \]  

(5)

of eqn. (3) exists nonsingular, constant matrix \( C \) such that

\[ \Phi(t + T) = \Phi(t)C \]  

(6)
Matrix $C$ is called a fundamental matrix of the solution $\Phi$. Fundamental matrix $\hat{C}$ for every other solution of equation (3) is similar to $C$, it can be obtained from $C$ by similarity transformation.

Constant matrix $B$ of eqn. (5) is bound with fundamental matrix $C$ as follows

$$e^{TB} = C$$

(7)

An eigenvalue $\lambda$ of fundamental matrix $C$ is called a characteristic number of equation (4), the number $\frac{1}{T} \ln \lambda$ is called a characteristic exponent of equation (3). Each solution $x(t)$ of equation (3) with periodical varying coefficients can be obtained from solution $y(t)$ of equation (4) with constant coefficients as

$$x(t) = S(t)y(t)$$

(8)

where transfer matrix $S(t)$ is $T$-periodic.

To make more comprehensive the terms used for linear periodically varying systems, we illustrate it with very simple, first order homogenous differential equation

$$\frac{dx}{dt} = -[a + v \sin(\omega_1 t)]x$$

(9)

where $a, v$ and $\omega_1$ are constants.

The particular solution of equation (11) can be written as

$$x = \Phi(t) = e^{-at} e^{-v \cos t}$$

(10)

Solution (10) fulfil the general property (6) containing the fundamental matrix.

III. FREQUENCY CHARACTERISTICS OF THE PERIODICALLY VARYING CONTINUOUS SYSTEM

It is possible to obtain frequency characteristics of a linear time invariant system by at least three approaches. When transmittance function $H(s)$ is given, then the frequency characteristic is obtained immediately by putting $s = j \omega$. Two other experiments lead to the same results.

Applying sinusoidal input function with varying frequency it is possible to get the amplitude and phase of the output function, which form the frequency characteristic. The frequency characteristic can be obtained also from impulse response function $h(t)$, taking Fourier transform $F(j \omega) = F[h(j \omega)]$.

For a system with constant coefficients each of this three approaches gives the same results. A different situation takes a place when the system is time varying. Different approaches give different results. It is difficult or impossible to express the transmittance of such a system by elementary functions. It is possible to obtain frequency characteristics applying harmonic function as the input function. Results obtained via sinusoidal input function and Dirac input function are different. Below these observations will be illustrated by simulation of example equation (9).

Let the system described by eqn. (9) be excited with sinusoidal input function. Differential equation of such excited system is

$$\frac{dx}{dt} = -[a + v \sin(\omega_1 t)]x + u(t)$$

(11)

where $u(t) = U \sin(\omega t)$.

The frequency $\omega$ of the input function will be changed and steady state solution $x(t)$ will be computed in order to get frequency characteristic of the system. Equation (13) with numerical values of coefficients $a = 1$, $v = 0.5$, $\omega_1 = 2\pi$ and $u = 1$ is solved for changing frequency $\omega$.

![Fig. 1. Sequence of $x(t)$ solutions of the varying system for changed frequency of sinusoidal input function](image)

Fig. 1 shows the sequence of solutions $x(t)$ when input frequency $f = \omega / 2\pi$ is changed from 0.25Hz to 1.25 Hz. Duration of the constant frequency is 20 seconds and then it is changed by 0.25 Hz. Duration of 20 seconds is enough to get steady state and make FFT transform, as it will be further shown.

![Fig. 2. Amplitude frequency characteristics obtained from harmonic response of the systems](image)

Putting in eqn. (113) $v = 0$ it is possible to obtained similar computation results for the associated non-varying system. The non-varying system gives sinusoidal response in a steady state, while the varying system response contain also
harmonics different from those contained in the input function.

The time functions shown in Fig. 1 were further processed in order to obtain frequency characteristics of the systems. The waveforms extracted from 8 seconds intervals ending each constant frequency duration are processed with the use of FFT transform. Such calculations were made for input frequency changing from 0 to 5 Hz. For each input frequency associated harmonic of the response is computed. The results are presented in Fig. 2. From this figure we can see, that amplitude characteristics both systems are almost identical.

The second approach is by applying impulse input function. Eqn. (17) is solved for input function equal to Dirac function $u(t) = \delta(t)$. Fig. 3 shows the impulse responses of two systems: constant and varying. The simulation results were obtained for two values of coefficient $\nu = 0.5$ and $\nu = 0$. Of course, the impulse responses of two systems are not identical. The impulse response of time varying system depends on time instant when Dirac function is applied.

The Fourier transforms of these time responses are shown in Fig. 4. The 8 seconds wide of time window was chosen for computing FFT transform.

IV. FREQUENCY CHARACTERISTICS OF THE PERIODICALLY VARYING DISCRETE SYSTEM

The $z$-domain description of the t-v systems allows us to treat quickly tuned systems in which no inertia typical to their analog counterparts exists. That makes deeper insight into time variant phenomena in its pure form.

Let consider the t-v example of the lowpass, tuned cut-off frequency system in which $N=100$ linear phase FIR weighting is tuned each new sample with the period of 4 samples (Fig. 5).

![Fig. 5. Periodically varying example discrete system instant frequency characteristics.](image)

The frequency domain illustration of the system properties as given in Fig 5 is virtual, as for quickly tuned systems the instant frequency responses are too short and can not be directly observable. They manifest in reality as local instant time domain FIR signal processing in which the output/input relations fulfill the equation (14):

$$y(n) = \sum x(n-k)h_{FIR}(k,n \mod N) \quad (14)$$

where the $h_{FIR}(k,l)$ is the (n by m) FIR response matrix of the periodically varying LTV system. One can easily show that for the considered system the set of m different, position dependent, impulse responses can be observable (eqn. 15):

$$h_{imp}(k,l) = h_{FIR}(k,(k + l) \mod N) \quad (15)$$

Fig. 3. Impulse responses of time varying and time invariant systems

Fig. 4. Amplitude frequency characteristics obtained from impulse responses
The physical impulse responses \( h_{imp} \) lose the symmetry and have quite different frequency properties then \( h_{FIR} \) (Fig. 2).

There is a periodicity in the observable impulse response of the system - each \( m \) samples the \( h_{imp}(k,l) \) response passes through (samples) the same \( h_{FIR}(k,l) \) response (Fig. 2). That property is explained by Eqn. (7), (12), and exists for analog system (Fig. 3).

In spite of a clear time domain physical meaning the \( h_{imp} \) response frequency properties do not explain the behavior of the time-varying system (Fig. 7). They are equivalents of Fig 4 analog system data, however no resemblance to Fig 5 (time invariant) data appears.

The input signal is still convoluted with the varying \( h_{imp} \), but no level/phase canceling of undersampled signal false frequency components takes place as it is in the LTI case. For the LTI and LTV periodic systems, the properly sampled (Nyquist) input signal can be decomposed into the set of \( m \) undersampled components which are filtered by time invariant \( h_{imp} \) filters. However only for the LTI systems the superposition preserves the nonpolluted, but shaped signal spectra. For the LTV case the sinusoidal excitations answer, in general, with nonsinusoidal output waveforms (Fig. 8).

The location of false, leakage spectral components is known, but their level is signal and filter dependent and not easy to predict in the general case.

For the considered case the sinusoidal tests answer with a pure sinusoidal output for low frequency excitations \( (f_t<1*f_p) \), so that for that frequency range no time variability is observable. The high frequency excitations do not pass trough the system \( (f_t>0.3*f_p) \). Such observations cannot be easily explained with Fig. 7 data, but are obvious when filter realization (Eqn. 14) and Fig 1 plots are taken into account.

Each of the FIR weighting responses with zero output for any samples pattern of sinusoidal waveforms with the frequency higher than \( f_p \). On the other side, any low frequency signal component with \( f_s<0.1f_p \) answers with the same output for any filter from the Fig 1. That means that low frequency signal components are not, and should not be affected by the time variance in the discussed example. The testing signals spectra presented in Fig. 9 illustrate that discussion.

In the practical conclusion appears there: some specific behavior of \( h_{imp} \) frequency responses can explain the LTV filtering properties. Time variance affects only time-variant spectral regions of the signal working band. Signals having spectral components in that variant spectral bands answer with polluted spectra. The details of that spectra

REFERENCES